

EXTREMAL MULTIPLIERS OF THE DRURY-ARVESON SPACE

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ABSTRACT. We give a new characterization of the so-called *quasi-extreme* multipliers of the Drury-Arveson space H_d^2 , and show that every quasi-extreme multiplier is an extreme point of the unit ball of the multiplier algebra of H_d^2 .

1. INTRODUCTION

In [8] and [9] we introduced the notion of a *quasi-extreme* multiplier of the Drury-Arveson space H_d^2 , and gave a number of equivalent formulations of this property. (The relevant definitions are recalled in Section 2.) The main motivation is that in one variable, each of these conditions is equivalent to b being an extreme point of the unit ball of H^∞ (the space of bounded analytic functions in the unit disk $\mathbb{D} \subset \mathbb{C}$). The purpose of this paper is to give one further characterization of quasi-extremity in the general case, from which it will follow that every quasi-extreme multiplier of H_d^2 is in fact an extreme point of the unit ball of the multiplier algebra $\mathcal{M}(H_d^2)$. (The converse statement, namely whether or not every extreme point is quasi-extreme in our sense, remains open.) In particular we will prove the following theorem:

Theorem 1.1. *A contractive multiplier b of H_d^2 is quasi-extreme if and only if the only multiplier a satisfying*

$$(1.1) \quad M_a^* M_a + M_b^* M_b \leq I$$

is $a \equiv 0$.

Since the corollary concerning extreme points follows immediately, we prove it here:

Corollary 1.2. *If $b \in \text{ball}(\mathcal{M}(H_d^2))$ is quasi-extreme, then b is an extreme point of $\text{ball}(\mathcal{M}(H_d^2))$.*

Proof. We prove the contrapositive. If b is not extreme, then there exists a nonzero $a \in \text{ball}(\mathcal{M}(H_d^2))$ such that both $b \pm a$ lie in $\text{ball}(\mathcal{M}(H_d^2))$, that is, are contractive multipliers of H_d^2 . We then have the operator inequalities

$$M_{b+a}^* M_{b+a} \leq I, \quad M_{b-a}^* M_{b-a} \leq I$$

averaging these inequalities gives

$$M_a^* M_a + M_b^* M_b \leq I$$

so by Theorem 1.1, b is not quasi-extreme. □

The remainder of the paper is devoted to proving Theorem 1.1. Since the techniques required are rather different, the two implications of the theorem will be proved as two separate propositions, Propositions 4.1 and 5.1. In Section 2 we recall the Drury-Arveson space and

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its multipliers, and review the necessary results concerning the de-Branges Rovnyak type spaces $\mathcal{H}(b)$ contractively contained in H_d^2 , and in particular the solutions to the Gleason problem in these spaces. We define the *quasi-extreme* multipliers and review some equivalent formulations of this property that will be used later. In Section 3 we study the non-quasi-extreme multipliers in more detail, and extend to this class of functions some of the results proved by Sarason [11] in the one-variable case. Our results rely heavily on the construction of a particular solution to the Gleason problem with good extremal properties, which is carried out in this section. In Sections 4 and 5 respectively we prove Propositions 4.1 and 5.1.

2. THE DRURY-ARVESON SPACE, MULTIPLIERS, AND QUASI-EXTREMITY

The *Drury-Arveson space* is the Hilbert space of holomorphic functions defined on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$ with reproducing kernel

$$k_w(z) = k(z, w) = \frac{1}{1 - zw^*}; \quad z, w \in \mathbb{B}^d$$

(Here we use the notation $z = (z_1, z_2, \dots, z_d)$, so that $zw^* = \sum_{j=1}^d z_j \overline{w_j}$.) General facts about the H_d^2 spaces may be found in the recent survey [13].

A holomorphic function b on \mathbb{B}^d will be called a *multiplier* if $bf \in H_d^2$ whenever $f \in H_d^2$. In this case the operator $M_b : f \rightarrow bf$ is bounded, and we let $\mathcal{M}(H_d^2)$ denote the Banach algebra of multipliers, equipped with the operator norm. (Warning: the multiplier norm always dominates the supremum norm of b over the unit ball, but the two are in general unequal. Also, not every bounded function b is a multiplier, see *e.g.* [13, 1].) For the reproducing kernel k_w we have $M_b^* k_w = b(w)^* k_w$. It follows that $\|M_b\| \leq 1$ if and only if the expression

$$k_w^b(z) = k^b(z, w) := \frac{1 - b(z)b(w)^*}{1 - zw^*},$$

defines a positive kernel on \mathbb{B}^d . When this is the case we let $\mathcal{H}(b)$ denote the associated reproducing kernel Hilbert space, called the deBranges-Rovnyak space of b . The space $\mathcal{H}(b)$ is a space of holomorphic functions on \mathbb{B}^d , contained in H_d^2 , and the inclusion map $\mathcal{H}(b) \subset H_d^2$ is contractive for the respective Hilbert space norms. We write $\|\cdot\|_b$ and $\langle \cdot, \cdot \rangle_b$ for the norm and inner product in $\mathcal{H}(b)$ respectively.

Properties of the spaces $\mathcal{H}(b)$ when $d > 1$ were studied in [8], inspired among other things by the results of Sarason in the one variable case [11],[12]. In one variable, the $\mathcal{H}(b)$ spaces are invariant under the backward shift; in several variables we instead (following Ball, Bolotnikov, and Fang [2]) consider solutions to the *Gleason problem*: given a function $f \in \mathcal{H}(b)$, we seek functions $f_1, \dots, f_d \in \mathcal{H}(b)$ such that

$$(2.1) \quad f(z) - f(0) = \sum_{j=1}^d z_j f_j(z).$$

From [2] we know that this problem always has a solution; in fact there exist (not necessarily unique) bounded operators X_1, \dots, X_d acting on $\mathcal{H}(b)$ such that the functions $f_j := X_j f$ solve (2.1 for any $f \in \mathcal{H}(b)$. Moreover these X_j can be chosen to be *contractive* in the following

sense: for every $f \in \mathcal{H}(b)$,

$$(2.2) \quad \sum_{j=1}^d \|X_j f\|_b^2 \leq \|f\|_b^2 - |f(0)|^2.$$

These contractive solutions were studied further in [8], where we proved the following (see also [9] for the vector-valued case):

Proposition 2.1. *A set of bounded operators (X_1, \dots, X_d) is a contractive solution to the Gleason problem in $\mathcal{H}(b)$ if and only if the X_j act on reproducing kernels by the formula*

$$(2.3) \quad X_j k_w^b = w_j^* k_w^b - b(w)^* b_j$$

for some choice of functions $b_1, \dots, b_d \in \mathcal{H}(b)$ which satisfy

- (i) $\sum_{j=1}^d z_j b_j(z) = b(z) - b(0)$,
- (ii) $\sum_{j=1}^d \|b_j\|_b^2 \leq 1 - |b(0)|^2$.

The set all contractive solutions X is in one-to-one correspondence with the set of all tuples b_1, \dots, b_d satisfying these conditions [9, Theorem 4.10]. We will call such sets of b_j *admissible*, or say that such a set is a *contractive Gleason solution* for b .

It turns out that for some contractive multipliers b , the operators X_j of the proposition are unique, that is, there is only one admissible tuple. When this happens we will call the multiplier b *quasi-extreme*. (The original definition of quasi-extreme in [8] is different, involving the so-called noncommutative Aleksandrov-Clark state for b , but this definition will be easier to work with for the present purposes.) In [8] and [9] we gave a number of equivalent formulations of quasi-extremity, we recall only a few of them here.

Proposition 2.2. *Let b be a contractive multiplier of H_d^2 . The following are equivalent:*

- i) b is quasi-extreme.
- ii) There is a unique contractive solution (X_1, \dots, X_d) to the Gleason problem in $\mathcal{H}(b)$.
- iii) There exists a contractive solution (X_1, \dots, X_d) such that the equality $\sum_{j=1}^d \|X_j f\|_b^2 = \|f\|_b^2 - |f(0)|^2$ holds for every $f \in \mathcal{H}(b)$.
- iv) There is a unique admissible tuple (b_1, \dots, b_d) satisfying the conditions of Proposition 2.1.
- v) All admissible tuples (b_1, \dots, b_d) are extremal, i.e.

$$\sum_{j=1}^d \|b_j\|_b^2 = 1 - |b(0)|^2,$$

for any admissible tuple.

- vi) $\mathcal{H}(b)$ does not contain the function b .
- vii) $\mathcal{H}(b)$ does not contain the constant functions.

In [9] these equivalences were extended to the case of operator-valued b .

What will be most useful in what follows is item (v); in particular b is *not* quasi-extreme if and only if there exists an admissible tuple (b_1, \dots, b_d) which obeys the strict inequality

$$\sum_{j=1}^d \|b_j\|_b^2 < 1 - |b(0)|^2.$$

3. NON-QUASI-EXTREME b

Let b denote a contractive multiplier of the Drury-Arveson space H_d^2 on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$. We assume throughout that $b(z)$ is not constant. We let $G_b(z)$ denote the Cayley transform or Herglotz-Schur function of b :

$$(3.1) \quad G_b(z) = \frac{1 + b(z)}{1 - b(z)}$$

and we construct the reproducing kernel Hilbert spaces $\mathcal{H}(b)$, $\mathcal{L}(b)$, the deBranges-Rovnyak and Herglotz spaces of b , respectively, with the kernels

$$(3.2) \quad k_w^b(z) := \frac{1 - b(z)b(w)^*}{1 - zw^*},$$

and,

$$(3.3) \quad K_w^b(z) := \frac{1}{2} \frac{G_b(z) + G_b(w)^*}{1 - zw^*} = (1 - b(z))^{-1}(1 - b(w)^*)^{-1}k_w^b(z).$$

The map $f \rightarrow (1 - b)f$ is thus a unitary multiplier from $\mathcal{L}(b)$ onto $\mathcal{H}(b)$.

We define an operator $V : \mathcal{L}(b)^d \rightarrow \mathcal{L}(b)$ by declaring

$$(3.4) \quad V \begin{pmatrix} z_1^* K_z^b \\ \vdots \\ z_d^* K_z^b \end{pmatrix} := K_z^b - K_0^b$$

on the span of the the columns appearing in the definition, and defining V to be 0 on the orthogonal complement of this span. A quick calculation using the formula for the reproducing kernel (3.3) shows that

$$zw^* K_w^b(z) = \langle K_w^b - K_0^b, K_z^b - K_0^b \rangle_{\mathcal{L}(b)} \quad \text{for all } z, w \in \mathbb{B}^d$$

and hence that V is a partial isometry. It follows that V^* is 0 on the orthogonal complement of the set $\{K_z^b - K_0^b : z \in \mathbb{B}^d\}$. Note that a vector $f \in \mathcal{L}(b)$ is orthogonal to this set if and only if $f(z) = f(0)$ for all z ; that is, if and only if f is constant. (In particular V is a coisometry if and only if the only constant function in $\mathcal{L}(b)$ is 0.) We next observe:

Lemma 3.1. *The space $\mathcal{L}(b)$ contains the constants if and only if $\mathcal{H}(b)$ contains b ; that is, by Proposition 2.2, if and only if b is not quasi-extreme.*

Proof. $1 \in \mathcal{L}(b)$ if and only if $1 - b \in \mathcal{H}(b)$; since $k_0^b = 1 - b(z)b(0)^* \in \mathcal{H}(b)$ always, the lemma follows. \square

For the remainder of this section we assume that b is not quasi-extreme, so by the lemma, $\mathcal{L}(b)$ contains the constant functions. By construction the tuple (V_1, \dots, V_d) is a row contraction and $\sum_{j=1}^d V_j V_j^*$ is the projection in $\mathcal{L}(b)$ orthogonal to the constants; so that $V_j^* 1 = 0$ for all j . We first record some facts about the V_j that will be of use later.

From the definition of V we have

$$(3.5) \quad V_j^*(K_z - K_0) = z_j^* K_z^b.$$

We also record the following chain of equalities for later use; these use only the fact that $V_j^*1 = 0$: For each j ,

$$\begin{aligned}
V_j^*(K_0) &= V_j^* \frac{1}{2} \left(\frac{2}{1-b} - 1 + \frac{1+b(0)^{**}}{1-b(0)} \right) \\
&= V_j^* \left(\frac{1}{1-b} - 1 \right) \\
(3.6) \quad &= V_j^* \left(\frac{b}{1-b} \right).
\end{aligned}$$

We next observe that the V_j solve the Gleason problem in $\mathcal{L}(b)$; indeed for $f \in \mathcal{L}(b)$ we take the inner product of f with the identity

$$(3.7) \quad K_z^b - K_0^b = \sum_{j=1}^d z_j^* V_j K_z^b$$

we get

$$(3.8) \quad f(z) - f(0) = \sum_{j=1}^d z_j (V_j^* f)(z).$$

We can now define operators S_j on $\mathcal{H}(b)$ conjugate to the V_j via the unitary $g \rightarrow \frac{1}{\sqrt{2}}(1-b)g$; specifically for $g \in \mathcal{H}(b)$ we define

$$(3.9) \quad (S_j^* g)(z) = (1-b) V_j^* \frac{g}{1-b}.$$

Again the row $S = (S_1, \dots, S_d)$ is a row contraction; in fact a row partial isometry whose final space $\text{ran}(\sum_{j=1}^d S_j S_j^*)$ is the orthogonal complement of the one-dimensional space spanned by $1-b$.

We now use the operators S_j to define an admissible tuple b_1, \dots, b_d and construct the associated solution to the Gleason problem in $\mathcal{H}(b)$ with good extremal properties. In particular put

$$(3.10) \quad b_j = (1-b(0)) S_j^* b \in \mathcal{H}(b)$$

and define operators X_j as in (2.3).

Proposition 3.2. *The tuple $X = (X_1, \dots, X_d)$ is a contractive solution to the Gleason problem in $\mathcal{H}(b)$. Moreover it is the unique solution with the property such that*

$$(3.11) \quad X_j b = b_j$$

where the b_j are those belonging to X_j .

Proof. The fact that V^b can be used to define a contractive Gleason solution for $K(b)$ in this way is a special case of [9, Theorem 4.4, Lemma 4.6]. We include a proof below for completeness.

We first verify that the b_j defined by (3.10) are admissible. Since the V_j^* solve the Gleason problem in $\mathcal{L}(b)$, we have

$$\begin{aligned}\sum_{j=1}^d z_j V_j^* \left(\frac{b}{1-b} \right) &= \frac{b(z)}{1-b(z)} - \frac{b(0)}{1-b(0)} \\ &= \frac{b(z) - b(0)}{(1-b(z))(1-b(0))}\end{aligned}$$

so by the definition of S_j^* and b_j

$$\begin{aligned}b(z) - b(0) &= (1-b(0))(1-b(z)) \sum_{j=1}^d z_j (V_j^* \frac{b}{1-b})(z) \\ &= (1-b(0)) \sum_{j=1}^d z_j S_j^* b(z) \\ &= \sum_{j=1}^d z_j b_j(z).\end{aligned}$$

To prove the norm inequality, observe that

$$\begin{aligned}\sum_{j=1}^d \|b_j\|_b^2 &= \sum_{j=1}^d \|S_j^* b\|_b^2 \\ &= |1-b(0)|^2 \sum_{j=1}^d \|V_j^* \frac{b}{1-b}\|_{\mathcal{L}(b)}^2 \\ &= |1-b(0)|^2 \sum_{j=1}^d \|V_j^* K_0\|_{\mathcal{L}(b)}^2 \quad \text{by (3.6)} \\ &\leq 1 - |b(0)|^2.\end{aligned}$$

where the last inequality holds since V^* is a column contraction and $\|K_0\|^2 = \frac{1-|b(0)|^2}{|1-b(0)|^2}$. Moreover, we observe that, since V^* is a partial isometry, equality holds in the above chain if and only if K_0 is orthogonal to the scalars, but this obviously never happens, so the inequality is always strict in this case when V^b is not a co-isometry. This also shows that this choice of admissible b_j minimizes the sum $\sum_{j=1}^d \|b_j\|_b$ over all choices of admissible b_j (see also [9, Corollary 4.8, Remark 4.9]).

To show that $X_j b = b_j$, we first show that

$$(3.12) \quad X_j = S_j^* - (1-b(0))^{-1} b_j \otimes k_0^b.$$

This equation follows from Clark-type intertwining formulas of [9, Section 4.15].

Indeed, from (3.6) we have

$$V_j^* K_0 = V_j^* \left(\frac{1}{1-b} \right) = \frac{1}{1-b} S_j^* b = \frac{1}{(1-b)(1-b(0))} b_j$$

The formula (3.12) is then verified by checking it on kernels k_w^b , where we have from the definition of S_j^*

$$\begin{aligned}
S_j^* k_w^b &= (1-b)V_j^* \left(\frac{1-bb(w)^*}{(1-b)(1-zw^*)} \right) \\
&= (1-b)(1-b(w)^*)V_j^*(K_w^b) \\
&= (1-b)(1-b(w)^*)V_j^*(K_w - K_0 + K_0) \\
&= w_j^* k_w^b + \frac{1-b(w)^*}{1-b(0)} b_j
\end{aligned}$$

and so

$$\begin{aligned}
S_j^* k_w^b - [(1-b(0))^{-1} b_j \otimes k_0^b] k_w^b &= w_j^* k_w^b + \frac{1-b(w)^*}{1-b(0)} b_j - \frac{1}{1-b(0)} (1-b(0)b(w)^*) b_j \\
&= w_j^* k_w^b - b(w)^* b_j \\
&= X_j k_w^b
\end{aligned}$$

as desired.

Finally, the claim that $X_j b = b_j$ follows immediately from (3.12) and the definition of the b_j in (3.10). \square

Remark: We observe in passing that these X_j annihilate the scalars: indeed, from the definition of X_j in (2.3) and the fact that $X_j b = b_j$, we have

$$X_j 1 = X_j (1 - b(0)^* b + b_0^* b) = X_j k_0^b + b(0)^* X_j b = -b(0)^* b_j + b(0)^* b_j = 0.$$

We also have that the defect operator $I - \sum X_j^* X_j$ has rank two when b is non-extreme:

Proposition 3.3. *Let b be a non-extreme multiplier. If X_j is a solution to the Gleason problem in $\mathcal{H}(b)$ with $\sum \|b_j\|^2 = 1 - |b(0)|^2 - |a_0|^2$, then*

$$(3.13) \quad I - \sum X_j^* X_j = k_0^b \otimes k_0^b + |a_0|^2 b \otimes b.$$

Proof. We first compute the inner product $\langle X_j^* X_j k_w^b, k_z^b \rangle$, using (2.3):

$$\begin{aligned}
\langle X_j^* X_j k_w^b, k_z^b \rangle &= \langle X_j k_w^b, X_j k_z^b \rangle \\
&= \langle w_j^* k_w^b - b(w)^* b_j, z_j^* k_z^b - b(z)^* b_j \rangle \\
&= z_j w_j^* k^b(z, w) - z_j b_j(z) b(w)^* - w_j^* b_j(w)^* b(z) + \|b_j\|_b^2 b(z) b(w)^*.
\end{aligned}$$

Summing over $j = 1, \dots, d$ (and using the fact that the b_j are admissible) gives

$$\sum_{j=1}^d \langle X_j^* X_j k_w^b, k_z^b \rangle = zw^* k^b(z, w) - (b(z) - b(0)) b(w)^* - b(z) (b(w)^* - b(0)^*) + (1 - |b(0)|^2 - |a_0|^2) b(z) b(w)^*.$$

Finally, we find

$$\begin{aligned}
\langle (I - \sum_{j=1}^d X_j X_j^*) k_w^b, k_z^b \rangle &= (1 - zw^*) k^b(z, w) + (b(z) - b(0)) b(w)^* + b(z) (b(w)^* - b(0)^*) \\
&\quad - (1 - |b(0)|^2 - |a_0|^2) b(z) b(w)^* \\
&= 1 - b(z) b(0)^* - b(0) b(w)^* + b(0) b(z) b(w)^* b(0)^* + |a_0|^2 b(z) b(w)^* \\
&= \langle (k_0^b \otimes k_0^b + |a_0|^2 b \otimes b) k_w^b, k_z^b \rangle.
\end{aligned}$$

This completes the proof. \square

Since b is assumed non-constant, this proposition shows that the range of $I - \sum_{j=1}^d X_j^* X_j$ is two dimensional, spanned by $k_0 = 1 - b(0)^* b$ and b , or equivalently, by b and 1. We can use this fact and the foregoing identity to relate the “defect” $|a_0|^2 := 1 - |b(0)|^2 - \sum_{j=1}^d \|b_j\|_b^2$ of the admissible tuple b_1, \dots, b_d to the $\mathcal{H}(b)$ -norm of the function b (compare [11],[3] for the scalar and vector valued cases, respectively, in one variable).

Lemma 3.4. *Suppose b is not quasi-extreme, and let (X_1, \dots, X_j) and b_1, \dots, b_j be as in Proposition 3.2. Define $a_0 > 0$ by $|a_0|^2 = 1 - |b(0)|^2 - \sum_{j=1}^d \|b_j\|_b^2$. Then*

$$(3.14) \quad |a_0|^2 = \frac{1}{1 + \|b\|_b^2}.$$

Proof. We compute $(I - \sum_{j=1}^d X_j^* X_j) b$ in two different ways. First, from its definition, and using the fact that $X_j b = b_j$,

$$(I - \sum_{j=1}^d X_j^* X_j) b = b - \sum_{j=1}^d X_j^* b_j.$$

Then observe that

$$\begin{aligned}
\sum_{j=1}^d (X_j^* b_j)(z) &= \sum_{j=1}^d \langle X_j^* b_j, k_z^b \rangle_b \\
&= \sum_{j=1}^d \langle b_j, z_j^* k_z^b - b_j b(z)^* \rangle_b \\
&= b(z) - b(0) - \sum_{j=1}^d \|b_j\|_b^2 b(z),
\end{aligned}$$

so that

$$(3.15) \quad (I - \sum_{j=1}^d X_j^* X_j) b = b(0) + b \sum_{j=1}^d \|b_j\|_b^2.$$

(Here we have used the fact that the X_j^* act by

$$(X_j^* f)(z) = z_j f(z) - \langle f, b_j \rangle_b b(z),$$

which follows easily from the definition of the X_j and the reproducing formula $(X_j^* f)(z) = \langle X_j^* f, k_z^b \rangle_b = \langle f, X_j k_z^b \rangle_b$.) Second, using the defect formula (3.13),

$$(3.16) \quad (I - \sum_{j=1}^d X_j^* X_j) b = b(0) k_0^b + |a_0|^2 \|b\|_b^2 b = b(0) + (-|b(0)|^2 + |a_0|^2 \|b\|_b^2) b(z).$$

Equating (3.15) and (3.16) gives

$$b(0) + b(z) \sum_{j=1}^d \|b_j\|_b^2 = b(0) + (-|b(0)|^2 + |a_0|^2 \|b\|_b^2) b(z)$$

Subtracting $b(0)$ from both sides leaves an equality between two constant multiples of $b(z)$; since b is assumed nonzero we have

$$-|b(0)|^2 + |a_0|^2 \|b\|_b^2 = \sum_{j=1}^d \|b_j\|_b^2 = 1 - |b(0)|^2 - |a_0|^2$$

and solving for $|a_0|^2$ gives (3.14). □

4. THE a -FUNCTION

In this section we prove the first half of Theorem 1.1:

Proposition 4.1. *If b is not quasi-extreme, then there exists a nonzero multiplier a such that*

$$M_a^* M_a + M_b^* M_b \leq I.$$

In the one-variable case if b is not extreme then there is an outer function a defined by the property that

$$(4.1) \quad |a(\zeta)|^2 + |b(\zeta)|^2 = 1 \quad \text{a.e. on } \mathbb{T};$$

we can assume that $a(0) > 0$. In the above \mathbb{T} denotes the unit circle. For this a we have immediately $M_a^* M_a + M_b^* M_b = I$. It is known in general that an equality of this sort cannot hold when $d > 1$ except in trivial cases (where the functions are constant), see [6]. In any case, when $d > 1$ we do not have any direct recourse to the theory of outer functions so different methods are required.

Nonetheless, the proof of (4.1) is in a sense constructive: a will be given in terms of a *transfer function realization* [4],[2]. It is remarkable that the algebraic construction given here, if carried out in one variable, produces exactly the outer function in (4.1). This follows from our transfer function realization and Sarason's computation of the Taylor coefficients of a [11]; we prove this at the end of the section.

We begin by recalling the relevant facts about transfer function realizations [4] and the *generalized functional models* of [2].

Let $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ be Hilbert spaces and let \mathcal{X}^d denote the direct sum of d copies of \mathcal{X} . By a *d-colligation* we mean an operator $\mathbf{U} : \mathcal{X} \oplus \mathcal{U} \rightarrow \mathcal{X}^d \oplus \mathcal{Y}$ expressed in the block matrix form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}$$

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The colligation is called *contractive, isometric, unitary, etc.* if \mathbf{U} is an operator of that type. For points $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, it will be convenient to identify z with the row contraction:

$$z : \mathcal{X}^d \rightarrow \mathcal{X}; \quad z \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} := z_1 x_1 + \dots + z_d x_d.$$

Observe that $\|z\|^2 = \|zz^*\|_{\mathcal{L}(\mathcal{X})} = \sum_{j=1}^d |z_j|^2$, so $\|z\| = |z|_{\mathbb{C}^d} < 1$ if and only if $z \in \mathbb{B}^d$. If \mathbf{U} is a contractive colligation, the *transfer function* for \mathbf{U} is

$$S(z) = D + C(I - zA)^{-1}zB.$$

The transfer function $S(z)$ is a holomorphic function in \mathbb{B}^d taking values in the space of bounded operators from \mathcal{U} to \mathcal{Y} . (For our purposes we will only need to consider finite-dimensional \mathcal{U} and \mathcal{Y}). It is a theorem of Ball, Trent and Vinnikov [4] that b is a contractive multiplier of $H_d^2 \otimes \mathcal{U}$ into $H_d^2 \otimes \mathcal{Y}$ if and only if it possesses a transfer function realization. In [2], it was shown that such a transfer function could always be chosen to be of a special form, called a *generalized functional model* realization. In particular, (in the case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$) if $X = (X_1, \dots, X_d)$ is a contractive solution to the Gleason problem in $\mathcal{H}(b)$, if we take $\mathcal{X} = \mathcal{H}(b)$ and define for all $f \in \mathcal{H}(b)$ and $\lambda \in \mathbb{C}$

- $A_j f = X_j f, j = 1, \dots, d$
- $B_j \lambda = \lambda b_j, j = 1, \dots, d$
- $Cf = f(0)$
- $D\lambda = b(0)\lambda$

then the corresponding colligation is contractive and its transfer function is $b(z)$. Since $Cf = f(0) = \langle f, k_0^b \rangle$, we will write k_0^{b*} for C and express the colligation as

$$\mathbf{U} = \begin{bmatrix} X_1 & b_1 \\ \vdots & \vdots \\ X_d & b_d \\ k_0^{b*} & b(0) \end{bmatrix}$$

Proof of Proposition 4.1. Fix the admissible tuple (b_1, \dots, b_d) and corresponding operators (X_1, \dots, X_d) of Proposition 3.2. We can then consider the colligation acting between $\mathcal{H}(b) \oplus \mathbb{C}$ and $\mathcal{H}(b) \oplus \mathbb{C}^2$ given by

$$\tilde{\mathbf{U}} = \begin{bmatrix} X_1 & b_1 \\ \vdots & \vdots \\ X_d & b_d \\ k_0^{b*} & b(0) \\ -a_0 b^* & a_0 \end{bmatrix}$$

We claim that $\tilde{\mathbf{U}}$ is isometric. If this is so, then the colligation

$$\mathbf{V} = \begin{bmatrix} X_1 & b_1 \\ \vdots & \vdots \\ X_d & b_d \\ -a_0 b^* & a_0 \end{bmatrix}$$

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is contractive, and hence the associated transfer function is a contractive multiplier $a(z)$. Moreover it is apparent that a is nonzero, since $a(0) = a_0 \neq 0$.

With a defined in this way, the transfer function associated to $\tilde{\mathbf{U}}$ is the 2×1 multiplier

$$\begin{pmatrix} b \\ a \end{pmatrix}$$

which is contractive; this proves the proposition.

It remains to prove the claim that $\tilde{\mathbf{U}}$ is isometric. Let us write out $\tilde{\mathbf{U}}^* \tilde{\mathbf{U}}$ explicitly; we have

$$\tilde{\mathbf{U}}^* \tilde{\mathbf{U}} = \begin{bmatrix} \sum_{j=1}^d X_j X_j^* + k_0^b \otimes k_0^b + |a_0|^2 b \otimes b & \sum_{j=1}^d X_j^* b_j + b(0) k_0^b - |a_0|^2 b \\ * & \sum_{j=1}^d \|b_j\|_b^2 + |b(0)|^2 + |a_0|^2 \end{bmatrix}$$

where we note that $(2, 1)$ entry is just the adjoint of the $(1, 2)$ entry. We consider the entries of the right-hand side one at a time.

The $(1, 1)$ entry is equal to the identity operator on $\mathcal{H}(b)$ by (3.13).

The $(2, 2)$ entry is equal to 1 by the definition of a_0 in Proposition 3.3.

The $(1, 2)$ (and by symmetry, $(2, 1)$) entry is equal to 0. To see this we use again the fact that $X_j b = b_j$ and compute:

$$\begin{aligned} \sum_{j=1}^d X_j^* b_j &= \sum_{j=1}^d [z_j b_j(z) - b(z) \|b_j\|_b^2] \\ &= -b(0) + b(z) \left(1 - \sum_{j=1}^d \|b_j\|_b^2\right) \\ &= -b(0) + (|a_0|^2 + |b(0)|^2) b(z) \\ &= -b(0) (1 - b(0)^* b(z)) + |a_0|^2 b(z) \\ &= -b(0) k_0^b + |a_0|^2 b(z) \end{aligned}$$

Thus $\tilde{\mathbf{U}}$ is isometric, which finishes the proof. □

4.1. The one-variable case. We analyze the foregoing construction in the one-variable case. Here the Drury-Arveson space becomes the classical Hardy space $H^2(\mathbb{D})$ and its multiplier algebra is the space of bounded analytic functions $H^\infty(\mathbb{D})$, equipped with the supremum norm. In this case it is known [12] that $b \in \text{ball}(H^\infty)$ is quasi-extreme if and only if it is an extreme point of $\text{ball}(H^\infty)$, which is equivalent to the condition

$$(4.2) \quad \int_{\mathbb{T}} \log(1 - |b|^2) dm = -\infty.$$

(See [7, p.138]). Conversely, if b is not (quasi)-extreme, this integral is finite, and hence there exists (as noted at the beginning of this section) an outer function $a \in \text{ball}(H^\infty)$ satisfying

$$(4.3) \quad |a(\zeta)|^2 + |b(\zeta)|^2 = 1$$

for almost every $|\zeta| = 1$; this a is unique if we impose the normalization $a(0) > 0$.

In this setting, there is of course ever only one solution to the Gleason problem in $\mathcal{H}(b)$, namely the usual backward shift operator on holomorphic functions

$$S^*f(z) = \frac{f(z) - f(0)}{z}.$$

Following Sarason [11] we denote the restriction

$$X = S^*|_b.$$

All of the above discussion of transfer function realizations applies here, so b is realized by the colligation

$$\mathbf{U} = \begin{bmatrix} X & S^*b \\ k_0^{b*} & b(0) \end{bmatrix}$$

Let now a be the outer function of (4.3) with $a(0) > 0$.

We expand a as a power series

$$a(z) = \sum_{n=0}^{\infty} \hat{a}(n) z^n.$$

Sarason [11] proves the following formula for the Taylor coefficients $\hat{a}(n)$:

Proposition 4.2. *We have $|a(0)|^2 = \frac{1}{1+\|b\|_b^2}$ and for $n \geq 1$*

$$\langle X^n b, b \rangle_{\mathcal{H}(b)} = \frac{-\hat{a}(n)}{a(0)}.$$

Mutlplying by z^n and summing we get

$$\begin{aligned} a(z) &= a(0) - a(0) \sum_{n=1}^{\infty} \langle X^n b, b \rangle_b z^n \\ &= a(0) - a(0) \langle (I - zX)^{-1} zXb, b \rangle_b \end{aligned}$$

Since $Xb = S^*b$, this shows that a is a transfer function for the colligation

$$\begin{bmatrix} X & S^*b \\ -a(0)b^* & a(0) \end{bmatrix}$$

acting on $\mathcal{H}(b) \oplus \mathbb{C}$. Finally, since

$$a(0) = \sqrt{\frac{1}{1 + \|b\|_b^2}} = a_0$$

(the first equality by Proposition 4.2 and the second by Lemma 3.4) this is precisely the transfer function which is used to define a in the proof of Proposition 4.1.

5. CONCLUSION OF THE PROOF OF THEOREM 1.1

In this section we prove the second half of Theorem 1.1:

Proposition 5.1. *If b is a multiplier of H_d^2 and there exists a nonzero multiplier a such that*

$$M_a^* M_a + M_b^* M_b \leq I$$

then b is not quasi-extreme.

The proof requires an elementary-seeming lemma, which nonetheless appears easiest to prove using the notion of a *free lifting* of a multiplier. We review the relevant results, prove the lemma, and finally prove Proposition 5.1.

We recall quickly the construction of the *free or non-commutative Toeplitz algebra* of Popescu. This is a canonical example of a *free semigroup algebra* as described by Davidson and Pitts [5], which contains proofs of all the claims made here. Fix an alphabet of d letters $\{1, \dots, d\}$ and let \mathbb{F}_d^+ denote the set of all words w in these d letters, including the empty word \emptyset . The set \mathbb{F}_d^+ is a saemigroup under concatenation: if $w = i_1 \dots i_n$ and $v = j_1 \dots j_m$, we define

$$wv = i_1 \dots i_n j_1 \dots j_m.$$

Let F_d^2 denote the Hilbert space (called the *Fock space*) with orthonormal basis $\{\xi_w\}_{w \in \mathbb{F}_d^+}$. This space comes equipped with a system of isometric operators L_1, \dots, L_d which act on basis vectors ξ_w by *left creation*:

$$L_i \xi_w = \xi_{iw}.$$

The operators L_1, \dots, L_d obey the relations

$$L_i^* L_j = \delta_{ji} I,$$

in other words they are isometric with orthogonal ranges. The *free semigroup algebra* \mathcal{L}_d is the WOT-closed algebra of bounded operators on F_d^2 generated by L_1, \dots, L_d . Each operator $F \in \mathcal{L}_d$ has Fourier-like expansion

$$(5.1) \quad F \sim \sum_{w \in \mathbb{F}_d^+} f_w L^w$$

where, for a word $w = i_1 \dots i_n$, by L^w we mean the product $L_{i_1} L_{i_2} \dots L_{i_n}$. The coefficients f_w are determined by the relation

$$f_w = \langle F \xi_\emptyset, \xi_w \rangle_{F_d^2}$$

and the Cesaro means of the series converge WOT to F . To each $F \in \mathcal{L}_d$ we can associated a d -variable holomorphic function $\lambda(F)$ as follows: to each word $w = i_1 \dots i_n$ let z^w denote the product

$$z^w = z_{i_1} z_{i_2} \dots z_{i_n}.$$

(Observe that $z^w = z^v$ precisely when w is obtained by permuting the letters of v). Then for $F \in \mathcal{L}_d$ we define $\lambda(F)$ by the series

$$\lambda(F)(z) = \sum_{w \in \mathbb{F}_d^+} f_w z^w.$$

The series converges uniformly on compact subsets of \mathbb{B}^d , and is always a multiplier of H_d^2 . In fact, Davidson and Pitts prove that the map λ is completely contractive from \mathcal{L}_d to $\mathcal{M}(H_d^2)$. Conversely, if $f \in \mathcal{M}(H_d^2)$ and $\|f\| \leq 1$, then there exists (by commutant lifting) an $F \in \mathcal{L}_d$ (not necessarily unique) such that $\|F\| \leq 1$ and $\lambda(F) = f$. We call such an F a *free lifting* of f . Free liftings also always exist for matrix-valued multipliers, so in particular if, say,

$$\begin{pmatrix} f \\ g \end{pmatrix}$$

is a contractive 2×1 multiplier, then there exist $F, G \in \mathcal{L}_d$ such that $\lambda(F) = f, \lambda(G) = g$, and

$$\begin{pmatrix} F \\ G \end{pmatrix}$$

is contractive.

We will need the following lemma, which we prove using free liftings:

Lemma 5.2. *If b is a multiplier and there exists a nonzero multiplier a satisfying $M_a^* M_a + M_b^* M_b \leq I$, then an a can be chosen satisfying this inequality and such that $a(0) \neq 0$.*

Proof. By the above remarks there exist free liftings A, B of a and b to the free semigroup algebra \mathcal{L}_d such that the column $\begin{pmatrix} B \\ A \end{pmatrix}$ is contractive. The element A has Fourier expansion

$$A \sim \sum a_w L^w$$

with $a_\emptyset = 0$ (since $a(0) = 0$). Choose a word v of minimal length such that $c_v \neq 0$. It follows that

$$\tilde{A} = L_v^* A = \sum_w c_w L_v^* L_w = \sum_u \tilde{c}_u L_u$$

is a contractive free multiplier, and $\tilde{A}(0) := \tilde{c}_\emptyset = c_v \neq 0$, and we then have that

$$\begin{pmatrix} B \\ \tilde{A} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & L_v^* \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}$$

is contractive. Since the Davidson-Pitts symmetrization map λ is completely contractive, on putting $\tilde{a} = \lambda(\tilde{A})$ we have $\tilde{a}(0) \neq 0$ and

$$\begin{pmatrix} b \\ \tilde{a} \end{pmatrix}$$

is a contractive 2×1 multiplier, which proves the lemma. \square

Remark: This is really the same proof that works in the disk (without the need for the free lifting step). In the disk we just get that \tilde{a} satisfies $a(z) = z^n \tilde{a}(z)$ for some n , and hence

$$M_a^* M_a = M_{\tilde{a}}^* M_{\tilde{a}};$$

(since M_z is an isometry). More generally we could let $a = \theta F$ be the inner-outer factorization of a ; since M_θ is isometric we would have

$$M_a^* M_a = M_F^* M_\theta^* M_\theta M_F = M_F^* M_F.$$

Proof of Proposition 5.1. Suppose that b is a contractive multiplier and there exists a nonzero multiplier a so that

$$M_a^* M_a + M_b^* M_b \leq I$$

By the lemma we may assume that $a(0) \neq 0$. We will construct an admissible tuple b_1, \dots, b_d such that

$$\sum_{j=1}^d \|b_j\|_b^2 \leq 1 - |b(0)|^2 - |a(0)|^2 < 1 - |b(0)|^2;$$

by the remark following Proposition 2.2 this proves that b is not quasi-extreme.

Let

$$c = \begin{pmatrix} b & 0 \\ a & 0 \end{pmatrix}.$$

Then c is a 2×2 contractive multiplier, and

$$(5.2) \quad c(0)^* c(0) = \begin{pmatrix} |b(0)|^2 + |a(0)|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

We form the deBranges-Rovnyak space $\mathcal{H}(c)$ of the function c , which has reproducing kernel

$$\begin{aligned} k^c(z, w) &= \frac{I - c(z)c(w)^*}{1 - zw^*} \\ &= \begin{bmatrix} k^b(z, w) & \frac{-b(z)a(w)^*}{1-zw^*} \\ \frac{-a(z)b(w)^*}{1-zw^*} & k^a(z, w) \end{bmatrix}. \end{aligned}$$

Now we apply the vector-valued generalization of a basic result from the theory of reproducing kernel Hilbert spaces: let $H(k)$ be a \mathcal{H} -valued RKHS of functions on a set X . An \mathcal{H} -valued function F on X belongs to $H(k)$ if and only if there is a $t \geq 0$ such that

$$F(x)F(y)^* \leq t^2 k(x, y),$$

as positive $\mathcal{L}(\mathcal{H})$ -valued kernel functions on X . Moreover the least such t that works is $t = \|F\|_{H(k)}$ [10, Theorem 10.17].

Note that in the above we view $F(x) : \mathbb{C} \rightarrow \mathcal{H}$ as a linear map for any fixed $x \in X$. It follows that $F(y)^* h = \langle F(y), h \rangle_{\mathcal{H}}$ for any $h \in \mathcal{H}$. For example, if (as in the case of $\mathcal{H}(c)$) $\mathcal{H} = \mathbb{C}^2$ then in the standard basis $F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$ and

$$F(x)F(y)^* = \begin{bmatrix} F_1(x)\overline{F_1(y)} & F_1(x)\overline{F_2(y)} \\ F_2(x)\overline{F_1(y)} & F_2(x)\overline{F_2(y)} \end{bmatrix}.$$

So now let $C : \mathbb{C}^2 \rightarrow K(c) \otimes \mathbb{C}^d$ be a contractive Gleason solution for c , e.g., the one appearing in a generalized functional model realization for c (which exists by [2]): that is,

$$C = \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix} \text{ obeys}$$

$$zC(z) = z_1 c_1(z) + \dots + z_d c_d(z) = c(z) - c(0),$$

and contractivity means:

$$C^* C \leq I - c(0)^* c(0).$$

So each $c_j(z) \in \mathbb{C}^{2 \times 2}$ and we write

$$c_j(z) = \begin{bmatrix} b_j(z) & * \\ a_j(z) & * \end{bmatrix},$$

and observe that the $B = \begin{bmatrix} b_1 \\ \vdots \\ b_d \end{bmatrix}$, and the $A = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}$ are Gleason solutions for b, a in the sense that

$$b(z) - b(0) = \sum_{j=1}^d z_j b_j(z),$$

and similarly for a . Note that

$$c_j(z)e_1 = \begin{bmatrix} b_j(z) \\ a_j(z) \end{bmatrix}.$$

We need to check that B actually belongs to $\mathcal{H}(b) \otimes \mathbb{C}^d$ and is a contractive Gleason solution for b : Let $\{e_1, e_2\}$ denote the standard orthonormal basis of \mathbb{C}^2 and let $t_j := \|c_j e_1\|_{\mathcal{H}(c)}$. Then by the vector-valued RKHS proposition discussed above, and the form of the reproducing kernel for $\mathcal{H}(c)$,

$$\begin{aligned} (c_j(z)e_1)(c_j(w)e_1)^* &= \begin{bmatrix} b_j(z) \\ a_j(z) \end{bmatrix} \begin{bmatrix} b_j(w)^* & a_j(w)^* \end{bmatrix} \\ &= \begin{bmatrix} b_j(z)b_j(w)^* & b_j(z)a_j(w)^* \\ a_j(z)b_j(w)^* & a_j(z)a_j(w)^* \end{bmatrix} \\ &\leq t_j^2 \begin{bmatrix} k^b(z, w) & \frac{-b(z)a(w)^*}{1-zw^*} \\ \frac{-a(z)b(w)^*}{1-zw^*} & k^a(z, w) \end{bmatrix}, \end{aligned}$$

as positive kernel functions. In particular the $(1, 1)$ entry of the above equation must be a positive kernel function so that

$$b_j(z)b_j(w)^* \leq t_j^2 k^b(z, w).$$

Again, by the scalar version of the RKHS result this implies that $b_j \in \mathcal{H}(b)$ and that

$$\|b_j\|_{\mathcal{H}(b)} \leq t_j = \|c_j e_1\|_{\mathcal{H}(c)}.$$

This yields the inequalities

$$\begin{aligned} \sum_{k=1}^d \|b_j\|_{\mathcal{H}(b)}^2 &\leq \sum_{k=1}^d t_j^2 \\ &= \sum_{k=1}^d \|c_j e_1\|_{\mathcal{H}(c)}^2 \\ &= \sum_{k=1}^d \langle c_j^* c_j e_1, e_1 \rangle_{\mathbb{C}^2} \\ &= \langle C^* C e_1, e_1 \rangle_{\mathbb{C}^2} \\ &\leq \langle (I - c(0)^* c(0)) e_1, e_1 \rangle_{\mathbb{C}^2} \\ &= 1 - |b(0)|^2 - |a(0)|^2, \\ &< 1 - |b(0)|^2, \end{aligned}$$

and the proof is complete. □

REFERENCES

- [1] W.B. Arveson. Subalgebras of C^* -algebras. *Acta Math.*, 123:141–224, 1969.
- [2] Joseph A. Ball, Vladimir Bolotnikov, and Quanlei Fang. Schur-class multipliers on the Fock space: de Branges-Rovnyak reproducing kernel spaces and transfer-function realizations. In *Operator theory, structured matrices, and dilations*, volume 7 of *Theta Ser. Adv. Math.*, pages 85–114. Theta, Bucharest, 2007.
- [3] Joseph A. Ball and Thomas L. Kriete, III. Operator-valued Nevanlinna-Pick kernels and the functional models for contraction operators. *Integral Equations Operator Theory*, 10(1):17–61, 1987.
- [4] Joseph A. Ball, Tavan T. Trent, and Victor Vinnikov. Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces. In *Operator theory and analysis (Amsterdam, 1997)*, volume 122 of *Oper. Theory Adv. Appl.*, pages 89–138. Birkhäuser, Basel, 2001.
- [5] Kenneth R. Davidson and David R. Pitts. Nevanlinna-Pick interpolation for non-commutative analytic Toeplitz algebras. *Integral Equations Operator Theory*, 31(3):321–337, 1998.
- [6] Kunyu Guo, Junyun Hu, and Xianmin Xu. Toeplitz algebras, subnormal tuples and rigidity on reproducing $\mathbb{C}[z_1, \dots, z_d]$ -modules. *J. Funct. Anal.*, 210(1):214–247, 2004.
- [7] Kenneth Hoffman. *Banach spaces of analytic functions*. Dover Publications, Inc., New York, 1988. Reprint of the 1962 original.
- [8] Michael T. Jury. Clark theory in the Drury-Arveson space. *J. Funct. Anal.*, 266(6):3855–3893, 2014.
- [9] Michael T. Jury and Robert T.W. Martin. Aleksandrov-clark theory for the Drury-Arveson space. *preprint*.
- [10] V. Paulsen and M. Raghupathi. *An Introduction to the theory of reproducing kernel Hilbert spaces*. Cambridge Studies in Advanced Mathematics, 2016.
- [11] Donald Sarason. Shift-invariant spaces from the Brangesian point of view. In *The Bieberbach conjecture (West Lafayette, Ind., 1985)*, volume 21 of *Math. Surveys Monogr.*, pages 153–166. Amer. Math. Soc., Providence, RI, 1986.
- [12] Donald Sarason. *Sub-Hardy Hilbert spaces in the unit disk*. University of Arkansas Lecture Notes in the Mathematical Sciences, 10. John Wiley & Sons Inc., New York, 1994. A Wiley-Interscience Publication.
- [13] Orr Shalit. Operator theory and function theory in Drury-Arveson space and its quotients. <http://adsabs.harvard.edu/abs/2013arXiv1308.1081S>.

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